

JOURNAL OF APPROXIMATION THEORY 37, 293–310 (1983)

# $L_\infty$ -Upper Bound of $L_2$ -Projections onto Splines at a Geometric Mesh

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Received March 17, 1982

For an integer  $k \geq 1$  and a geometric mesh  $(q^i)_{i \in \mathbb{Z}}$  with  $q \in (0, \infty)$ , let

$$M_{i,k}(x) := k[q^i, \dots, q^{i+k}](\cdot - x)_+^{k-1},$$

$$N_{i,k}(x) := (q^{i+k} - q^i) M_{i,k}(x)/k,$$

and let  $A_k(q)$  be the Gram matrix  $(\int M_{i,k} N_{j,k})_{i,j \in \mathbb{Z}}$ . It is known that  $\|A_k(q)^{-1}\|_\infty$  is bounded independently of  $q$ . In this paper it is shown that  $\|A_k(q)^{-1}\|_\infty$  is strictly decreasing for  $q$  in  $[1, \infty)$ . In particular, the sharp upper bound and lower bound for  $A_k(q)^{-1}$  are obtained:

$$2k - 1 \leq \|A_k(q)^{-1}\|_\infty \leq \left(\frac{\pi}{2}\right)^{2k} \left\{ \sum_{j \in \mathbb{Z}} (1 + 2j)^{-2k} \right\}^{-1}$$

for all  $q \in (0, \infty)$ .

## 1. INTRODUCTION

Let  $\mathbf{x} := (x_i)_{i \in \mathbb{Z}}$  be a strictly increasing biinfinite sequence with  $x_{\pm \infty} := \lim_{i \rightarrow \pm \infty} x_i$  and  $I := (x_{-\infty}, x_{+\infty})$ . Further, let

$$S := mS_{k,\mathbf{x}}(I) := \{f \in C^{k-2}(I) \cap L_\infty(I);$$

$$f|_{(x_i, x_{i+1})} \text{ is a polynomial of degree } < k\}$$

be the normed linear space of bounded polynomial splines of order  $k$  with breakpoint sequence  $\mathbf{x}$  and norm  $\|f\| := \sup_{x \in I} |f(x)|$ . We shall be concerned with  $P_S$ , the orthogonal projector onto  $S$  with respect to the ordinary inner product

$$(f, g) := \int_I f(x) g(x) dx,$$

\* Supported in part by the United States Army under Contract DAAG29-80-C-0041.

but restricted to  $L_\infty(I)$ . We want to bound its norm

$$\|P_S\|_\infty := \sup_{f \in L_\infty(I)} \|P_S f\| / \|f\|_\infty.$$

In 1973, de Boor raised the following

*Conjecture* [1].

$$\sup_{\mathbf{x}} \|P_S\|_\infty \leq \text{const}_k < \infty.$$

This conjecture has been verified for  $k = 1, 2, 3, 4$  (see de Boor [3] and the references cited there). De Boor [2] also obtained a bound of  $P_S$  in terms of a global mesh ratio. In general, however, this conjecture seems hard to solve. For a geometric mesh  $\mathbf{x}$ , Höllig [8] recently proved the boundedness of  $P_S$ . Later on, Feng and Kozak [6] reproved this result. Before recalling some results of theirs, we need to introduce some notations. For the mesh  $\mathbf{x} = (x_i)_{-\infty}^\infty$ , let

$$\begin{aligned} M_{i,k}(\mathbf{x}) &:= k[x_i, \dots, x_{i+k}](\cdot - x)_+^{k-1} \\ N_{i,k}(\mathbf{x}) &:= ([x_{i+1}, \dots, x_{i+k}] - [x_i, \dots, x_{i+k-1}])(\cdot - x)_+^{k-1} \\ &= (x_{i+k} - x_i) M_{i,k}(\mathbf{x}) / k. \end{aligned}$$

Set

$$A_k(i, j) := \int M_{i,k} N_{j,k} \quad \text{for } i, j \in \mathbb{Z}.$$

Let  $A_k \in \mathbb{R}^{\mathbb{Z} \times \mathbb{Z}}$  be the biinfinite matrix given by the rule

$$(i, j) \rightarrow A_k(i, j) \quad \text{for } (i, j) \in \mathbb{Z} \times \mathbb{Z}.$$

It was shown by de Boor [1] that

$$D_k^{-2} \|A_k^{-1}\|_\infty \leq \|P_S\|_\infty \leq \|A_k^{-1}\|_\infty,$$

where  $D_k$  is a constant depending only on  $k$ . Thus bounding  $P_S$  is equivalent to bounding  $A_k^{-1}$ .

Let us restrict ourselves now to a particular case where  $\mathbf{x}$  is a geometric mesh:  $\mathbf{x} := (q^i)_{-\infty}^\infty$  for some  $q \in [1, \infty)$  (note that the case  $q \in (0, 1]$  is symmetric to the case  $q \in [1, \infty)$ ; see [6, 8]). Spline interpolation at a geometric mesh was first investigated by Micchelli [9], who based his argument on the properties of the so-called generalized Euler–Frobenius polynomials. Later on, Feng and Kozak [6] developed such a consideration. Earlier, and in a different way, Höllig [8] made a more precise investigation into the boundedness of  $L_2$ -projections onto splines on a geometric mesh. In particular, he got the following elegant result (see [8, Theorem 5]):

**THEOREM A.** For a geometric mesh  $\mathbf{x} := (q^i)_{i=0}^\infty$  with  $q \in (0, \infty)$ , let  $A_k(q)$  be the biinfinite matrix  $(\int M_{i,k} N_{j,k})_{i,j \in \mathbb{Z}}$ . Then

$$\|A_k(q)^{-1}\|_\infty = |\Omega_k(q)|^{-1}, \quad (1)$$

where

$$\begin{aligned} \Omega_k(q) := & 2k! (k-1)! t^{2k-1} \prod_{v=1}^k \frac{q^v + 1}{q^v - 1} \prod_{v=1}^{k-1} \frac{q^v + 1}{q^v - 1} \\ & \times \sum_{j \in \mathbb{Z}} \prod_{v=1}^k \frac{1}{[\pi(1+2j)]^2 + (vt)^2} \end{aligned} \quad (2)$$

with  $t := \log q$ . Moreover,

$$\lim_{q \rightarrow 1} \Omega_k(q) = \left(\frac{2}{\pi}\right)^{2k} \left\{ \sum_{j \in \mathbb{Z}} (1+2j)^{-2k} \right\} \quad (3)$$

$$\lim_{q \rightarrow \infty} \Omega_k(q) = \frac{1}{2k-1}. \quad (4)$$

Based on numerical evidence, de Boor raised the following

*Conjecture.*  $\Omega_k(q)$  is a monotone increasing function on  $[1, \infty)$ .

This conjecture was verified for  $k \leq q$  by Feng and Kozak [7]. They also showed that  $\Omega_k(q) \leq 1/(2k-1)$  in the same paper.

The purpose of this paper is to confirm the above conjecture. Thus we have

**THEOREM 1.**  $\Omega_k(q)$  is a monotone increasing function on  $[1, \infty)$ . In particular,

$$2k-1 \leq \|A_k(q)^{-1}\|_\infty \leq \left(\frac{\pi}{2}\right)^{2k} \left\{ \sum_{j \in \mathbb{Z}} (1+2j)^{-2k} \right\}^{-1}. \quad (5)$$

Note that  $\Omega_1(q) \equiv 1$  and that  $\Omega_2(q) \equiv \frac{1}{3}$  in terms of a straightforward calculation. Hence we can restrict ourselves to the case  $k \geq 3$  from now on.

In Section 2, we shall give an alternative proof of Theorem A. Sections 3 and 4 will be devoted to proving the monotonicity of  $\Omega_k(q)$  for  $q \in [1, 20]$  and  $q \in [20, \infty)$ , respectively.

2. THE BOUND FOR  $A_k(q)^{-1}$ 

As before,  $\mathbf{x} = (q^l)_{-\infty}^{\infty}$  is a geometric mesh with  $q \in (1, \infty)$  and  $l = \log q$ . Consider

$$\phi_0(x) := [0, 1, \dots, (2k-1)]_z x^z / (q^z + q^k) \quad \text{for } x \in [1, q].$$

It is easy to verify that

$$\begin{aligned} q^l \phi_0^{(l)}(q) + q^k \phi_0^{(l)}(1) &= [0, 1, \dots, (2k-1)]_z \{z(z-1) \cdots (z-l+1)\} \\ &= 0, \quad \text{for } l = 1, \dots, 2k-2, \\ &= 1, \quad \text{for } l = 2k-1. \end{aligned} \quad (6)$$

Since  $\phi_0$  is a polynomial of degree  $2k-1$ ,  $\phi_0^{(2k-1)}$  is constant in  $[1, q]$ . Hence (6) yields that

$$\phi_0^{(2k-1)}(x) = 1/(q^k + q^{2k-1}) \quad \text{for } x \in [1, q]. \quad (7)$$

Now we extend the domain of  $\phi_0$  to  $(0, \infty)$  as

$$\phi(x) := (-q^k)^m \phi_0(q^{-m}x) \quad \text{for } q^m \leq x \leq q^{m+1}, \quad m \in \mathbb{Z}.$$

From (6) we assert that  $\phi \in \mathcal{S}_{2k, \mathbf{x}}$ , and that

$$\phi(q^m) = (-q^k)^m \phi_0(1), \quad m \in \mathbb{Z}. \quad (8)$$

It follows that

$$\begin{aligned} [x_0, x_1, \dots, x_{m-1}, x_m] \phi &= \frac{[x_1, \dots, x_m] \phi - [x_0, \dots, x_{m-1}] \phi}{x_m - x_0} \\ &= \frac{-q^{k-m+1} [x_0, \dots, x_{m-1}] \phi - [x_0, \dots, x_{m-1}] \phi}{q^m - 1} \\ &= -\frac{q^{k-m+1} + 1}{q^m - 1} [x_0, \dots, x_{m-1}] \phi. \end{aligned}$$

By induction on  $m$ , we can obtain

$$\begin{aligned} [x_0, x_1, \dots, x_k] \phi &= (-1)^k \left( \prod_{m=1}^k \frac{q^{k-m+1} + 1}{q^m - 1} \right) \phi_0(1) \\ &= (-1)^k \left( \prod_{m=1}^k \frac{q^m + 1}{q^m - 1} \right) \phi_0(1). \end{aligned} \quad (9)$$

From (8) we deduce that

$$[x_i, \dots, x_{i+k}] \phi = (-1)^i [x_0, \dots, x_k] \phi. \quad (10)$$

By Peano's theorem (see [4])

$$[x_i, \dots, x_{i+k}] \phi = \int M_{i,k}(x) \phi^{(k)}(x)/k! \, dx.$$

Now we get

$$\int M_{i,k}(x) \phi^{(k)}(x)/k! \, dx = (-1)^i (-1)^k \left( \prod_{m=1}^k \frac{q^m + 1}{q^m - 1} \right) \phi_0(1). \quad (11)$$

Obviously,  $\phi^{(k)}/k! \in \mathbb{S}_{k,x}$ ; hence  $\phi^{(k)}/k!$  may be expanded in a  $B$ -spline series

$$\phi^{(k)}/k! = \sum \alpha_j N_{j,k};$$

however,  $\phi^{(k)}(qx) = -\phi^{(k)}(x)$ . Thus

$$\sum \alpha_j N_{j,k}(x) = -\sum \alpha_j N_{j,k}(qx) = -\sum \alpha_j N_{j-1,k}(x) = -\sum \alpha_{j+1} N_{j,k}(x).$$

By the uniqueness of  $B$ -spline expansion we assert that

$$\alpha_{j+1} = -\alpha_j, \quad j \in \mathbb{Z}.$$

Thus we can write

$$\phi^{(k)}/k! = C \sum (-1)^j N_{j,k}, \quad (12)$$

where  $C$  is a constant to be determined. Now (11) and (12) together give

$$\sum_{j \in \mathbb{Z}} (-1)^j \int M_{i,k}(x) N_{j,k}(x) \, dx = (-1)^i C^{-1} (-1)^k \left( \prod_{m=1}^k \frac{q^m + 1}{q^m - 1} \right) \phi_0(1). \quad (13)$$

Let

$$\Omega_k(q) := C^{-1} (-1)^k \left( \prod_{m=1}^k \frac{q^m + 1}{q^m - 1} \right) \phi_0(1). \quad (14)$$

Then (see de Boor *et al.* [5])

$$\|A_k(q)^{-1}\|_{\infty} = |\Omega_k(q)|^{-1}.$$

It remains to determine  $C$ . Differentiate (12)  $k-1$  times,

$$\phi^{(2k-1)}/k! = C \left( \sum (-1)^j N_{j,k} \right)^{(k-1)}.$$

One the one hand,

$$\phi^{(2k-1)}(x)/k! = \frac{1}{k!} \frac{1}{q^k + q^{2k-1}} \quad \text{for } x \in (1, q).$$

On the other hand (see [4]),

$$\begin{aligned} & \left( \sum (-1)^j N_{j,k} \right)^{(k-1)} \\ &= \frac{2(k-1)}{q^{k-1}-1} \frac{(q+1)(k-2)}{q^{k-2}-1} \dots \frac{q^{k-2}+1}{q-1} \sum_{j \in \mathbb{Z}} (-1)^j (q^{k-1})^{-j} N_{j,1} \\ &= 2(k-1)! \frac{1}{q^{k-1}+1} \prod_{m=1}^{k-1} \frac{q^m+1}{q^m-1} \sum_{j \in \mathbb{Z}} (-1)^j (q^{k-1})^{-j} N_{j,1}. \end{aligned}$$

Thus, for  $x \in (1, q)$

$$\left( \sum (-1)^j N_{j,k} \right)^{(k-1)}(x) = 2(k-1)! \frac{1}{q^{k-1}+1} \prod_{m=1}^{k-1} \frac{q^m+1}{q^m-1}.$$

From the above calculation we get

$$\begin{aligned} C^{-1} &= k! (q^k + q^{2k-1}) 2(k-1)! \frac{1}{q^{k-1}+1} \prod_{m=1}^{k-1} \frac{q^m+1}{q^m-1} \\ &= 2k! (k-1)! q^k \prod_{m=1}^{k-1} \frac{q^m+1}{q^m-1}. \end{aligned} \quad (15)$$

Finally, (14) and (15) yield that

$$\begin{aligned} \Omega_k(q) &= (-1)^k 2k! (k-1)! \prod_{m=1}^k \frac{q^m+1}{q^m-1} \prod_{m=1}^{k-1} \frac{q^m+1}{q^m-1} q^k \phi_0(1) \\ &= (-1)^k 2k! (k-1)! \prod_{m=1}^k \frac{q^m+1}{q^m-1} \prod_{m=1}^{k-1} \frac{q^m+1}{q^m-1} \\ &\quad \times q^k [0, 1, \dots, 2k-1] \frac{1}{q + q^k}. \end{aligned} \quad (16)$$

We follow the procedure in [9] and use a well-known formula for the divided difference to get

$$\begin{aligned} & (-1)^k q^k [0, 1, \dots, 2k-1] \frac{1}{q + q^k} \\ &= \frac{(-1)^k q^k}{2\pi i} \left[ \int_{C_R} - \sum_{j=0}^{2k-1} \int_{C_{r_j}} \left( \prod_{m=0}^{2k-1} (z-m)(e^{z \log q} + q^k) \right)^{-1} dz \right], \end{aligned} \quad (17)$$

where  $C_R$  and  $C_{r_j}$  stand for positively oriented circles with centers at 0 and  $j$  and radius  $R$  and  $r_j$ , where  $R$  is sufficiently large and  $r_j$  sufficiently small,  $j = 0, 1, \dots, 2k-1$ .

Making  $R \rightarrow \infty$ ,  $r_j \rightarrow 0$  ( $j = 0, 1, \dots, 2k-1$ ) in (17) and using the residue theorem we get

$$\begin{aligned} & (-1)^k q^k [0, 1, \dots, 2k-1] \frac{1}{q + q^k} \\ &= (-1)^k q^k (-1) \sum_{j \in \mathbb{Z}} \operatorname{Res}_{z = i(\pi + 2\pi j)/t + k} \left( \prod_{v=0}^{2k-1} (z - v)(e^{z \log q} + q^k) \right)^{-1} \\ &= (-1)^k \sum_{j \in \mathbb{Z}} \left( t \prod_{v=0}^{2k-1} \left[ \frac{i(\pi + 2\pi j)}{t} - v + k \right] \right)^{-1} \\ &= \sum_{j \in \mathbb{Z}} t^{2k-1} \left/ \left( \prod_{v=0}^{2k-1} [\pi + 2\pi j - i(k - v)t] \right) \right. \\ &= t^{2k-1} \sum_{j \in \mathbb{Z}} \prod_{v=1}^k \frac{1}{(\pi + 2\pi j)^2 + (vt)^2}. \end{aligned}$$

Thus (2) is proved by substituting the above equality into (16). Then it is straightforward to verify (3). As to (4), we have

$$\begin{aligned} \lim_{q \rightarrow \infty} \Omega_k(q) &= (-1)^k 2k!(k-1)!/(2k-1)! \lim_{q \rightarrow \infty} \sum_{l=0}^{2k-1} (-1)^{l+1} \binom{2k-1}{l} \frac{q^k}{q^l + q^k} \\ &= 2(-1)^k \left/ \binom{2k-1}{k} \right/ \left[ \sum_{l=0}^{k-1} (-1)^{l+1} \binom{2k-1}{k} \right. \\ &\quad \left. + (-1)^{k+1} \frac{1}{2} \binom{2k-1}{k} \right] \\ &= 2(-1)^k \left/ \binom{2k-1}{k} \right/ \left\{ \sum_{l=0}^{k-1} (-1)^{l+1} \left[ \binom{2k-2}{l-1} + \binom{2k-2}{l} \right] \right. \\ &\quad \left. + (-1)^{k+1} \frac{1}{2} \binom{2k-1}{k} \right\} \\ &= 2(-1)^k \left/ \binom{2k-1}{k} \right/ \left[ (-1)^k \binom{2k-2}{k-1} + (-1)^{k+1} \frac{1}{2} \binom{2k-1}{k} \right] \\ &= \frac{2k}{2k-1} - 1 = \frac{1}{2k-1}. \end{aligned}$$

This ends the proof of Theorem A.

3. THE MONOTONICITY OF  $\Omega_k(q)$  FOR  $q \in [1, 20]$ 

Recall  $t = \log q$ . Let

$$f_k(t) := t^{2k-1} \prod_{v=1}^k \frac{e^{vt} + 1}{e^{vt} - 1} \prod_{v=1}^{k-1} \frac{e^{vt} + 1}{e^{vt} - 1} \sum_{j \in \mathbb{Z}} \prod_{v=1}^k \frac{1}{(\pi + 2\pi j)^2 + (vt)^2}.$$

Then  $\Omega_k(e^t) = 2k! (k-1)! f_k(t)$ . Consider  $f'_k(t)/f_k(t)$ . We have

$$\frac{f'_k(t)}{f_k(t)} = \sum_{j=0}^{\infty} \frac{1}{t} u_{k,j}(t) f_{k,j}(t), \quad (18)$$

where

$$\begin{aligned} u_{k,j}(t) &:= \prod_{v=1}^k \frac{1}{(\pi + 2\pi j)^2 + (vt)^2} \left/ \left\{ \sum_{l=0}^{\infty} \prod_{v=1}^k \frac{1}{(\pi + 2\pi l)^2 + (vt)^2} \right\} \right. \\ f_{k,j}(t) &:= 2k - 1 + \left( \sum_{v=1}^{k-1} + \sum_{v=1}^k \right) \left( \frac{vte^{vt}}{e^{vt} + 1} - \frac{vte^{vt}}{e^{vt} - 1} \right) \\ &\quad - \sum_{v=1}^k \frac{2(vt)^2}{(\pi + 2\pi j)^2 + (vt)^2}. \end{aligned} \quad (19)$$

If we can show that  $f'_k(t)/f_k(t) \geq 0$  for  $t \in [0, 3]$ , then  $\Omega'_k(q) \geq 0$  for  $q \in [1, 20]$ , because  $e^3 > 20$ . For this it suffices to show  $f_{k,0}(t) \geq 0$ , since  $f_{k,j}(t) \geq f_{k,0}(t)$  ( $j = 1, 2, \dots$ ) from (19). Let us first make the following observation:

**PROPOSITION 1.**

$$\frac{\pi^2}{\pi^2 + (cx)^2} \geq \frac{2xe^x}{e^{2x} - 1} \quad \text{for } x \in [0, \infty) \text{ and } c \in [1, 5/4].$$

*Proof.* Each of the following inequalities is equivalent to Proposition 1:

$$\begin{aligned} e^{2x} - 1 &\geq (1 + c^2 x^2 / \pi^2) 2xe^x, \\ \sum_{n=0}^{\infty} 2^n x^n / (n+1)! &\geq (1 + c^2 x^2 / \pi^2) \left( \sum_{n=0}^{\infty} x^n / n! \right), \\ \sum_{n=2}^{\infty} 2^n x^n / (n+1)! &\geq \sum_{n=2}^{\infty} \left[ \frac{1}{n!} + \frac{c^2}{\pi^2} \frac{1}{(n-2)!} \right] x^n. \end{aligned}$$

An induction argument on  $n$  shows, however, that

$$2^n / (n+1)! \geq \frac{1}{n!} + \frac{c^2}{\pi^2} \frac{1}{(n-2)!} \quad \text{for } n \geq 2 \text{ and } c \in [1, 5/4].$$

Therefore Proposition 1 is true.



PROPOSITION 2.

$$\frac{2\pi^2}{\pi^2 + (4x/3)^2} \geq \frac{\pi^2}{\pi^2 + (5x/4)^2} + \frac{\pi^2}{\pi^2 + (5x/3)^2}.$$

*Proof.*

$$\begin{aligned} & 2(\pi^2 + 25x^2/9)(\pi^2 + 25x^2/16) \\ &= 2\pi^4 + 1250\pi^2x^2/144 + 625x^2/72 \\ &\geq 2\pi^4 + 1137\pi^2x^2/144 + 625x^2/81 \\ &= (\pi^2 + 16x^2/9)[(\pi^2 + 25x^2/9) + (\pi^2 + 25x^2/16)]. \end{aligned}$$

Multiplying the above inequality by  $\pi^2/[(\pi^2 + \frac{16}{9}x^2)(\pi^2 + \frac{25}{16}x^2)]$ , we obtain Proposition 2.

PROPOSITION 3.

$$f_{k+1,0}(t) \geq f_{k,0}(t) \quad \text{for } t \geq 0 \text{ and } k \geq 3.$$

*Proof.* We shall argue by induction on  $k$ . For  $k = 3$ , we have

$$f_{4,0}(t) - f_{3,0}(t) = \frac{2\pi^2}{\pi^2 + (4t)^2} - \frac{2 \cdot 3te^{3t}}{e^{2 \cdot 3t} - 1} - \frac{2 \cdot 4te^{4t}}{e^{2 \cdot 4t} - 1}.$$

Set  $x := 3t$ . Then Propositions 1 and 2 yield that

$$\begin{aligned} f_{4,0}(t) - f_{3,0}(t) &\geq \left( \frac{\pi^2}{\pi^2 + (5x/4)^2} - \frac{2xe^x}{e^{2x} - 1} \right) \\ &\quad + \left( \frac{\pi^2}{\pi^2 + (5x/3)^2} - \frac{2(4xe^{4x/3}/3)}{e^{2 \cdot 4x/3} - 1} \right) \geq 0. \end{aligned}$$

Suppose now that  $k \geq 4$ . Then  $(k+1)/k \leq \frac{5}{4}$ . We have

$$\begin{aligned} f_{k+1,0}(t) - f_{k,0}(t) &= \frac{2\pi^2}{\pi^2 t [(k+1)t]^2} - \frac{2kte^{kt}}{e^{2kt} - 1} - \frac{2(k+1)te^{(k+1)t}}{e^{2(k+1)t} - 1} \\ &= \left( \frac{\pi^2}{\pi^2 + [(k+1)t]^2} - \frac{2kte^{kt}}{e^{2kt} - 1} \right) \\ &\quad + \left( \frac{\pi^2}{\pi^2 + [(k+1)t]^2} - \frac{2(k+1)te^{(k+1)t}}{e^{2(k+1)t} - 1} \right) \geq 0, \end{aligned}$$

according to Proposition 1. Thus Proposition 3 is proved.

Consequently,  $f_{k,j}(t) \geq f_{3,0}(t)$  for all  $k \geq 3$  and  $j \geq 0$ . The remaining task of this section is to elaborate the nonnegativity of  $f_{3,0}(t)$ . For this we need some estimates.

**PROPOSITION 4.** *Let  $h(x) = ((xe^x/(e^x + 1)) - \frac{1}{2}x)/x^2$ . Then  $h'(x) \leq 0$  for  $x \geq 0$ .*

*Proof.*  $h(x) = (e^x - 1)/2x(e^x + 1)$  and

$$h'(x) = \frac{1}{2} \frac{x(e^x + 1)e^x - (e^x - 1)[(e^x + 1) + xe^x]}{[x(e^x + 1)]^2} = \frac{1 + 2xe^x - e^{2x}}{2x^2(e^x + 1)^2},$$

while

$$\begin{aligned} 1 + 2xe^x - e^{2x} &= 1 + 2x \sum_{n=0}^{\infty} \frac{x^n}{n!} - \sum_{n=0}^{\infty} \frac{2^n x^n}{n!} \\ &= - \sum_{n=3}^{\infty} \frac{2(2^{n-1} - n)}{n!} x^n \leq 0 \quad \text{for } x \geq 0. \end{aligned}$$

**PROPOSITION 5.**

$$-xe^x/(e^x - 1) \geq -1 - \frac{1}{2}x - \frac{1}{12}x^2.$$

*Proof.*

$$\begin{aligned} &\left(1 + \frac{1}{2}x + \frac{1}{12}x^2\right)(e^x - 1) - xe^x \\ &= \sum_{n=2}^{\infty} \frac{x^{n+1}}{n!} - \sum_{n=1}^{\infty} \frac{1}{2} \frac{x^{n+1}}{n!} + \sum_{n=1}^{\infty} \frac{1}{12} \frac{x^{n+2}}{n!} \\ &= \sum_{n=5}^{\infty} \frac{1}{n!} \frac{n^2 - 7n + 12}{12} x^n \geq 0 \quad \text{for } x \geq 0. \end{aligned}$$

Now we are in a position to prove that  $f_{3,0}(t) \geq 0$  for  $t \in [0, 0.3]$ . Write

$$\begin{aligned} f_{3,0}(t) &= 2 \left(1 + \frac{te^t}{e^t + 1} - \frac{te^t}{e^t - 1} - \frac{t^2}{t^2 + \pi^2}\right) \\ &\quad + 2 \left(\frac{2te^{2t}}{e^{2t} + 1} - \frac{2te^{2t}}{e^{2t} - 1} - \frac{(2t)^2}{(2t)^2 + \pi^2}\right) \\ &\quad + \left(1 + \frac{3te^{3t}}{e^{3t} + 1} - \frac{3te^{3t}}{e^{3t} - 1} - \frac{(3t)^2}{(3t)^2 + \pi^2}\right) - \frac{9t^2}{\pi^2 + 9t^2}. \end{aligned}$$

It follows from Proposition 4 that, for  $t \in [0, 0.3]$ ,

$$\begin{aligned} te^t/(e^t + 1) - t/2 &\geq h(0.3)t^2 \geq 0.248t^2, \\ 2te^{2t}/(e^{2t} + 1) - 2t/2 &\geq h(0.6)(2t)^2 \geq 0.242(2t)^2, \\ 3te^{3t}/(e^{3t} + 1) - 3t/2 &\geq h(0.9)(3t)^2 \geq 0.234(3t)^2. \end{aligned}$$

In connection with Proposition 5, we obtain

$$\begin{aligned} 2 \left( 1 + \frac{te^t}{e^t + 1} - \frac{te^t}{e^t - 1} - \frac{t^2}{t^2 + \pi^2} \right) &\geq 2 \left( 0.248 - \frac{1}{12} - \frac{1}{\pi^2} \right) t^2 \\ &\geq 0.126t^2, \\ 2 \left( 1 + \frac{2te^{2t}}{e^{2t} + 1} - \frac{2te^{2t}}{e^{2t} - 1} - \frac{(2t)^2}{(2t)^2 + \pi^2} \right) &\geq 2 \left( 0.242 - \frac{1}{12} - \frac{1}{\pi^2} \right) (2t)^2 \\ &\geq 0.458t^2, \\ 1 + \frac{3te^{3t}}{e^{3t} + 1} - \frac{3te^{3t}}{e^{3t} - 1} - \frac{(3t)^2}{(3t)^2 + \pi^2} &\geq \left( 0.234 - \frac{1}{12} - \frac{1}{\pi^2} \right) (3t)^2 \\ &\geq 0.444t^2, \\ -\frac{9t^2}{\pi^2 + 9t^2} &\geq -\frac{9t^2}{\pi^2} \geq -0.912t^2. \end{aligned}$$

As a conclusion,  $f_{3,0}(t) \geq (0.126 + 0.458 + 0.444 - 0.912)t^2 = 0.116t^2$ . This shows that

$$f'_k(t) \geq 0 \quad \text{for } t \in [0, 0.3].$$

The next case we are going to treat is that of  $t \in [0.3, 3]$ . Let

$$\begin{aligned} v(t) &:= \frac{2\pi^2}{t^2 + \pi^2} + \frac{2\pi^2}{4t^2 + \pi^2} + \frac{\pi^2 - 9t^2}{\pi^2 + 9t^2}, \\ w(t) &:= \frac{4te^t}{e^{2t} - 1} + \frac{8te^{2t}}{e^{4t} - 1} + \frac{6te^{3t}}{e^{6t} - 1}. \end{aligned}$$

Then  $f_{3,0}(t) = v(t) - w(t)$ . It is easily seen that  $v'(t) \leq 0$  for  $t \in [0, \infty)$ . We claim that  $w'(t) \leq 0$  ( $0 \leq t < \infty$ ), too. This is guaranteed by

**PROPOSITION 6.** Let  $g(x) := 2xe^x/(e^{2x} - 1)$ . Then  $g'(x) \leq 0$  for  $x \geq 0$ .

*Proof.*  $g'(x) = -(2e^x/(e^{2x} - 1)^2)(1 + x + xe^{2x} - e^{2x})$ , while

$$\begin{aligned} 1 + x + xe^{2x} - e^{2x} &= (1 + x) - (1 - x) \sum_{n=0}^{\infty} \frac{2^n x^n}{n!} \\ &= \sum_{n=3}^{\infty} \frac{(n-2) 2^{n-1}}{n!} x^n \geq 0 \quad \text{for } x \geq 0. \end{aligned}$$

Accordingly,

$$f_{3,0}(t) = v(t) - w(t) \geq v(b) - w(a) \quad \text{for } t \in [a, b] \text{ with } 0 < a < b. \quad (20)$$

To determine the positivity of  $f_{3,0}$  we wrote a Fortran program and found that

$$v\left(\frac{n+1}{100}\right) - w\left(\frac{n}{100}\right) \geq 0.001 \quad \text{for } n = 30, 31, \dots, 299.$$

Thus by (20) we assert that

$$f_{3,0}(t) > 0 \quad \text{for } t \in \left[\frac{n}{100}, \frac{n+1}{100}\right], \quad n = 30, 31, \dots, 299.$$

Therefore

$$f_{3,0}(t) > 0 \quad \text{for } t \in \bigcup_{n=30}^{299} \left[\frac{n}{100}, \frac{n+1}{100}\right] = [0.3, 3].$$

So far we have shown that  $\Omega'_k(q) \geq 0$  for  $q \in [1, 20]$ .

#### 4. THE MONOTONICITY OF $\Omega_k(q)$ FOR $q \in [20, \infty)$

Let

$$f(q) := (-1)^k (2k-1)! q^k [0, 1, \dots, 2k-1] \frac{1}{q^{\cdot} + q^k}. \quad (21)$$

Then

$$\begin{aligned} f(q) &= (-1)^k q^k \sum_{l=0}^{2k-1} \binom{2k-1}{l} (-1)^{l+1} \frac{1}{q^l + q^k} \\ &= (-1)^k \sum_{l=0}^{2k-1} (-1)^{l+1} \binom{2k-1}{l} \frac{1}{1 + q^{l-k}} \end{aligned}$$

$$\begin{aligned}
&= \sum_{l=0}^{k-1} (-1)^{k+l+1} \binom{2k-1}{l} \frac{q^{k-l}}{1+q^{k-l}} \\
&\quad + (-1) \binom{2k-1}{k} \frac{1}{2} + \sum_{l=k+1}^{2k-1} (-1)^{k+l+1} \binom{2k-1}{l} \frac{1}{1+q^{l-k}}.
\end{aligned}$$

It follows that

$$\begin{aligned}
f'(q) &= \sum_{l=0}^{k-1} (-1)^{k+l+1} \binom{2k-1}{l} \frac{(k-l)q^{k-l-1}}{(1+q^{k-l})^2} \\
&\quad + \sum_{l=k+1}^{2k-1} (-1)^{k+l} \binom{2k-1}{l} \frac{(l-k)q^{l-k-1}}{(1+q^{l-k})^2} \\
&= \sum_{l=1}^k (-1)^{l-1} \binom{2k-1}{k-l} \cdot \frac{lq^{l-1}}{(1+q^l)^2} \\
&\quad - \sum_{l=1}^{k-1} (-1)^{l-1} \binom{2k-1}{k+l} \cdot \frac{lq^{l-1}}{(1+q^l)^2} \\
&= \sum_{l=1}^{k-1} (-1)^{l-1} \left[ \binom{2k-1}{k-l} - \binom{2k-1}{k+l} \right] \frac{lq^{l-1}}{(1+q^l)^2} \\
&\quad + (-1)^{k-1} \frac{kq^{k-1}}{(1+q^k)^2}. \tag{22}
\end{aligned}$$

Now we need the following propositions.

**PROPOSITION 7.**

$$\left[ \binom{2k-1}{k-l} - \binom{2k-1}{k+l} \right] \frac{lq^{l-1}}{(1+q^l)^2}$$

decreases as  $l$  increases and  $q \geq 6$ .

*Proof.*

$$\begin{aligned}
\binom{2k-1}{k-l} - \binom{2k-1}{k+l} &= \frac{(2k-1)!}{(k-l)!(k-1+l)!} \left(1 - \frac{k-l}{k+l}\right) \\
&= \frac{(2k-1)!}{(k-l)!(k+l)!} 2l. \tag{23}
\end{aligned}$$

We want to show

$$\begin{aligned}
&\frac{(2k-1)!}{(k-l)!(k+l)!} 2l \frac{lq^{l-1}}{(1+q^l)^2} \\
&\geq \frac{(2k-1)!}{(k-l-1)!(k+l+1)!} 2(l+1) \frac{(l+1)q^l}{(1+q^{l+1})^2} \quad \text{for } q \geq 6. \tag{24}
\end{aligned}$$

It is easily seen that (24) is equivalent to

$$\frac{1}{q} \left( \frac{q^{l+1} + 1}{q^l + 1} \right)^2 \geq \frac{k-l}{k+l+1} \left( 1 + \frac{1}{l} \right)^2;$$

however,

$$\frac{1}{q} \left( \frac{q^{l+1} + 1}{q^l + 1} \right)^2 = \frac{q^{2l+2} + 2q^{l+1} + 1}{q(q^{2l} + 2q^l + 1)} \geq q - 2q^{l-1} \geq q - 2,$$

because

$$\begin{aligned} (q - 2q^{1-l}) q(q^{2l} + 2q^l + 1) &= q^{2l+2} - q^2 - 2q^{2-l} \\ &\leq q^{2l+2} + 2q^{l+1} + 1. \end{aligned}$$

Meanwhile

$$\frac{k-l}{k+l+1} \left( 1 + \frac{1}{l} \right)^2 \leq 4.$$

Therefore (24) holds for  $q \geq 6$ , and Proposition 7 is proved.

**PROPOSITION 8.** For  $k \geq 2$  and  $q \geq 6$ ,

$$f'(q) \geq \binom{2k-1}{k-1} \frac{2}{k+1} \frac{1}{(1+q)^2} - \binom{2k-1}{k-2} \frac{4}{k+2} \frac{2q}{(1+q^2)^2}. \quad (25)$$

In particular,  $f'(q) \geq 0$  and

$$f(q) \leq \lim_{q \rightarrow \infty} f(q) = \binom{2k-1}{k-1} \frac{1}{2(2k-1)}. \quad (26)$$

*Proof.* Suppose first  $k$  is even,  $k = 2m$ . Then (22) and (23) yield that

$$\begin{aligned} f'(q) &= \binom{2k-1}{k-1} \frac{2}{k+1} \frac{1}{(1+q)^2} - \binom{2k+1}{k-2} \frac{4}{k+2} \frac{2q}{(1+q^2)^2} \\ &\quad + \sum_{j=2}^{m-1} \left[ \frac{(2k-1)!}{(k-2j+1)!(k+2j-2)!} \frac{2(2j-1)}{k+2j-1} \frac{(2j-1)q^{2j-2}}{(1+q^{2j-1})^2} \right. \\ &\quad \left. - \frac{(2k-1)!}{(k-2j)!(k+2j-1)!} \frac{2 \cdot 2j}{k+2j} \frac{2jq^{2j-1}}{(1+q^{2j})^2} \right] \\ &\quad + \left[ (2k-2) \frac{(k-1)q^{k-2}}{(1+q^{k-1})^2} - \frac{kq^{k-1}}{(1+q^k)^2} \right]. \end{aligned}$$

By Proposition 7 all the terms under the summation sign are positive. Moreover,

$$q^{k-2}/(1+q^{k-1})^2 \geq q^{k-1}/(1+q^k)^2 \quad \text{for } q \geq 1,$$

and

$$\begin{aligned} (2k-2)(k-1) \frac{q^{k-2}}{(1+q^{k-1})^2} - \frac{kq^{k-1}}{(1+q^k)^2} \\ \geq [(2k-2)(k-1) - k] \frac{q^{k-1}}{(1+q^k)^2} \geq 0 \quad \text{for } k \geq 2. \end{aligned}$$

For odd  $k$ , the proof is similar. Thus (25) holds. Furthermore,

$$\begin{aligned} f'(q) &\geq \frac{2}{(1+q)^2} \frac{(2k-1)!}{(k-2)!(k+2)!} \left[ \frac{k+2}{k-1} - \frac{4(1+1/q)^2}{q(1+1/q^2)^2} \right] \\ &\geq \frac{2}{(1+q)^2} \frac{(2k-1)!}{(k-2)!(k+2)!} \left[ 1 - 4 \left( 1 + \frac{1}{6} \right)^2 \right] \\ &\geq 0 \quad \text{for } q \geq 6, \end{aligned}$$

and

$$\begin{aligned} f(q) &\leq \lim_{q \rightarrow \infty} f(q) = \lim_{q \rightarrow \infty} \left[ \binom{2k-1}{k-1} \frac{1}{2} \Omega_k(q) \right] \\ &= \binom{2k-1}{k-1} \frac{1}{2(2k-1)}. \end{aligned}$$

PROPOSITION 9. *Let*

$$S(q) = 4 \sum_{v=1}^{k-1} \frac{vq^{v-1}}{q^{2v}-1} + 2 \frac{kq^{k-1}}{q^{2k}-1}.$$

*Then*

$$S(q) \leq \frac{4q^2}{(q^2-1)(q-1)^2} \quad \text{for } q \geq 1.$$

*Proof.* We have

$$S(q) = 4 \sum_{v=1}^{k-1} \frac{q^{2v}}{q^{2v}-1} vq^{-(v+1)} + 2kq^{-(k+1)} \frac{q^{2k}}{q^{2k}-1}.$$

Note that

$$\frac{q^{2v}}{q^{2v}-1} \leq \frac{q^2}{q^2-1} \quad \text{for } v \geq 1 \text{ and } q \geq 1.$$

Hence

$$\begin{aligned} S(q) &\leq \frac{4q^2}{q^2-1} \sum_{v=1}^{\infty} vq^{-(v+1)} = \frac{4q^2}{q^2-1} \frac{d}{dq} \left( - \sum_{v=1}^{\infty} q^{-v} \right) \\ &= \frac{4q^2}{q^2-1} \frac{d}{dq} \left( \frac{-1}{1-q^{-1}} \right) = \frac{4q^2}{(q^2-1)(q-1)^2}. \end{aligned}$$

PROPOSITION 10. *Let*

$$g_k(q) = \frac{2k-1}{k+1} \left[ 1 - \frac{4(k-1)}{k+2} \frac{q}{1+q^2} \right].$$

Then

$$g_{k+1}(q) \geq g_k(q) \quad k = 1, 2, 3, \dots, \quad q \geq 12.$$

*Proof.*

$$\begin{aligned} g_{k+1}(q) - g_k(q) &= \frac{3}{(k+1)(k+2)(k+3)} \left[ (k+3) - (3k-1) 4 \frac{q}{1+q^2} \right] \\ &\geq \frac{3}{(k+1)(k+1)(k+3)} \left[ (k+3) - (3k-1) 4 \frac{12}{1+12^2} \right] \\ &\geq 0 \quad \text{for } q \geq 12. \end{aligned}$$

Now we are in a position to prove the monotonicity of  $\Omega_k(q)$  for  $q \in [20, \infty)$ . From (16) and (21) we see that

$$\Omega_k(q) = 2k! (k-1)! / (2k-1)! \prod_{m=1}^k \frac{q^m+1}{q^m-1} \prod_{m=1}^{k-1} \frac{q^m+1}{q^m-1} f(q).$$

Hence

$$\frac{\Omega'_k(q)}{\Omega_k(q)} = - \left( 4 \sum_{v=1}^{k-1} \frac{vq^{v-1}}{q^{2v}-1} + 2 \frac{kq^{k-1}}{q^{2k}-1} \right) + \frac{f'(q)}{f(q)} = -S(q) + \frac{f'(q)}{f(q)}.$$

By Proposition 9 we have

$$S(q) \leq \frac{1}{q^2} \frac{4q^4}{(q^2-1)(q-1)^2} \leq \frac{1}{q^2} \frac{4 \cdot 20^4}{(20^2-1)(20-1)^2} \leq \frac{4.45}{q^2} \quad \text{for } q \geq 20.$$



Moreover, Propositions 8 and 10 tell us that

$$\begin{aligned}
 \frac{f'(q)}{f(q)} &\geq \frac{1}{(1+q)^2} \frac{4(2k-1)}{k+1} \left[ 1 - \frac{4(k-1)}{k+2} \frac{q}{1+q^2} \right] = \frac{4}{(1+q)^2} g_k(q) \\
 &\geq \frac{4}{(1+q)^2} g_4(q) = \frac{4}{(1+q)^2} \frac{7}{5} \left( 1 - \frac{2q}{1+q^2} \right) \\
 &= \frac{1}{q^2} \frac{28}{5} \frac{(1-1/q)^2}{(1+1/q)^2} \frac{1}{1+1/q^2} \\
 &\geq \frac{1}{q^2} \frac{28}{5} \frac{(1-(1/20))^2}{(1+(1/20))^2} \frac{1}{1+1/20^2} \\
 &\geq \frac{4.55}{q^2} \quad \text{for } q \geq 20 \text{ and } k \geq 4.
 \end{aligned}$$

Therefore

$$\begin{aligned}
 \frac{\Omega'_k(q)}{\Omega_k(q)} &= \frac{f'(q)}{f(q)} - S(q) \geq \frac{4.55}{q^2} - \frac{4.45}{q^2} = \frac{0.1}{q^2} > 0 \\
 &\quad \text{for } k \geq 4 \text{ and } q \geq 20.
 \end{aligned}$$

It remains to check the case  $k=3$ . For this we shall make a straightforward computation:

$$\begin{aligned}
 \Omega_3(q) &= 24 \left( \frac{q+1}{q-1} \right)^2 \left( \frac{q^2+1}{q^2-1} \right)^2 \frac{q^3+1}{q^3-1} q^3 (-1) [0, 1, 2, 3, 4, 5] \frac{1}{q^7+q^3} \\
 &= \frac{24}{120} \left( \frac{q+1}{q-1} \right)^2 \left( \frac{q^2+1}{q^2-1} \right)^2 \frac{q^3+1}{q^3-1} q^3 \left( \frac{1}{1+q^3} - 5 \frac{1}{q+q^3} \right. \\
 &\quad \left. + 10 \frac{1}{q^2+q^3} - 10 \frac{1}{2q^3} + 5 \frac{1}{q^3+q^4} - \frac{1}{q^3+q^5} \right) \\
 &= \frac{1}{5} \frac{q^2+1}{q^2+q+1}.
 \end{aligned}$$

Thus

$$\Omega'_3(q) = \frac{1}{5} \frac{q^2-1}{(q^2+q+1)^2} \geq 0 \quad \text{for } q \geq 1.$$

This completes the proof of Theorem 1.

## ACKNOWLEDGMENTS

I thank Professor Carl de Boor for his valuable help. Sections 2 and 4 especially reflect his suggestions.

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